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5 Magnetic Bioreactors

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5.1 Introduction

Currently tissue can be grown artificially outside the body in a bioreactor. In this process cells are seeded onto a biodegradable scaffold which is then inserted into the bioreactor. Cell growth is enabled by pumping a nutrient solution through the bioreactor. In practice tissue grown in this fashion frequently does not possess all the properties which are found in the corresponding naturally grown tissue. So for instance artificially grown bone does not possess the same strength as natural bone.

In the case of bone and cartilage the poor performance of artificially grown materials results from being grown in a stress free environment. A novel technique which is being developed to improve the properties of bone and cartilage grown *ex-vivo* is the so-called *magnetic force bioreactor*. In this device the cells are seeded with small magnetic particles before being placed in the bioreactor and, once *in situ*, forces are applied to the particles by the application of a magnetic field. It is known that the resulting stresses are capable of opening ion channels [6] and it is suspected that this is a crucial process necessary for normal development of the tissue.

The magnetic particles are manufactured from magnetite and are coated with a polymer to make them biocompatible. The diameters of the particles range in size from $10nm - 1\mu m$. Particles diameters larger than about $50nm$ behave as permanent magnets and are termed blocked whereas particles diameters smaller that this size have paramagnetic properties.

The aim of this report is twofold: firstly to determine the force or torque exerted on a particle of given size by a magnet and secondly to derive models for the transmission of stress in osteoblasts (bone cell) and chondrocytes (cartilage cell). The latter, we hope, will give a guide to whether ion channels on a cell open in response to a particular force. We note, though, that there are no models describing the the response of ion channels to mechanical stress.

5.2 The force on a magnetic particle

The properties of the magnetic particles used in this process depend greatly upon their size. Particles with diameter below about $50nm$ behave as superparamagnets, that is as a paramagnetic material but with large susceptibility χ, whereas particles with diameter much above $50nm$ behave as permanent magnetic dipoles.

The force acting on a (super)paramagnetic particle $\boldsymbol{F}_p$ in a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F}_p = \frac{\Upsilon \mathrm{Vol}}{2\mu_0} \nabla(|\boldsymbol{B}|^2). \tag{5.1}$$

Here Vol is the volume of the particle, μ_0 is the magnetic susceptibility of free space and Υ is a dimensionless number related to the magnetic susceptibility χ of the particle by the formula

$$\Upsilon = \frac{\chi}{1 + \mathcal{E}\chi},$$

where $\mathcal{E}$ is a shape constant, see for example [2]. It takes values 0 for a needle and 1/3 for a solid sphere. The susceptibility $\chi \gg 1$, so if we have a solid spherical particle, $\Upsilon \approx 3$. For elongated particles however, Υ can be large.

In contrast a particle with magnetic dipole behaviour is not only subject to a body force in a magnetic field, but also to a torque. Where the magnetic moment of the particle is $\boldsymbol{m}$ the torque $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \boldsymbol{m} \wedge \boldsymbol{B},$$

and the body force $\boldsymbol{F}_d$ by

$$\boldsymbol{F}_d = (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}.$$

The action of the torque $\boldsymbol{T}$ is such as to try and align the magnetic moment with the field. Since these magnetic particles are typically small in comparison with the lengthscale for variations in the field we expect the restoring force exerted on the edge of the particle required to oppose the torque to be much larger than the body force exerted on the particle.

We are asked to calculate the field for a disc magnet and estimate from this the magnitude of the force at different angles and distances from the magnet. Disc magnets usually come with magnetisation in the plane of the disc or perpendicular to it. The field in the latter case is relatively straightforward to calculate since, for a thin disc with uniform magnetisation, the field is the same as that for a circular current carrying wire and, where we rescale lengths with the radius of the disc L, takes the form

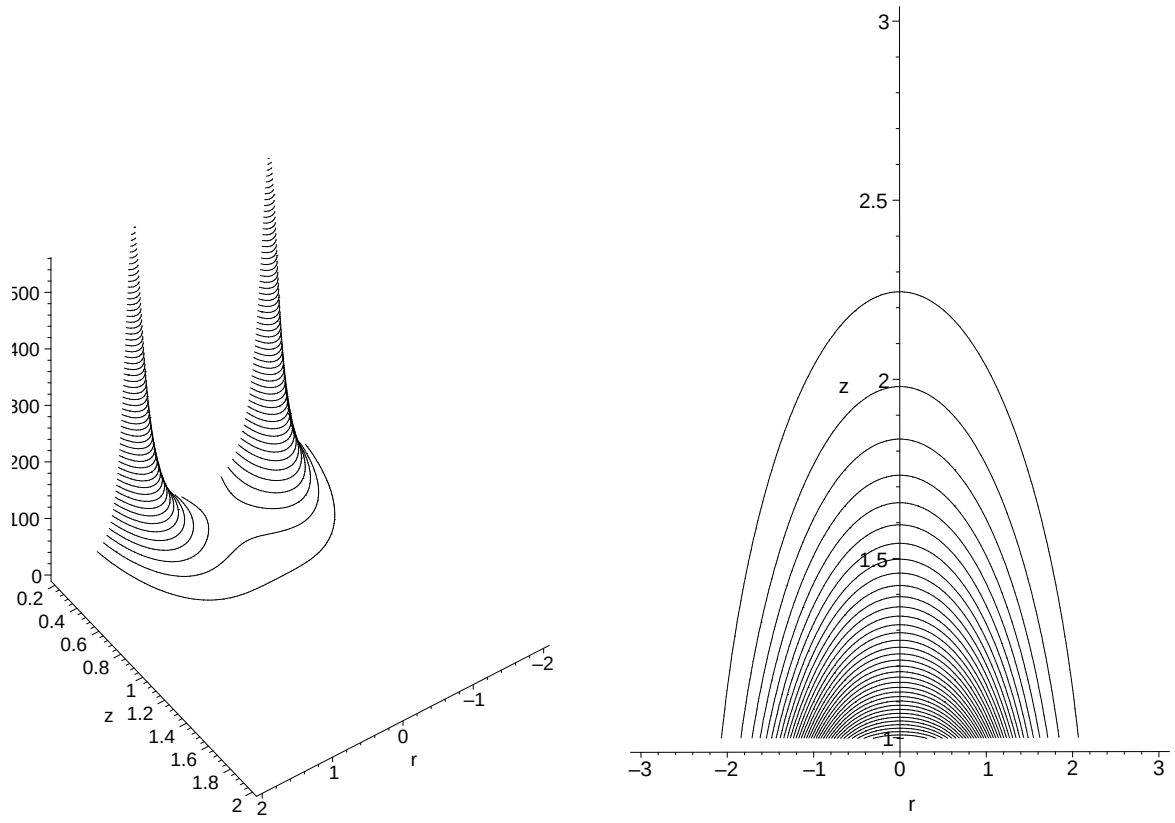
$$\boldsymbol{B} = B_r(r,z)\boldsymbol{e}_r + B_z(r,z)\boldsymbol{e}_z,$$

where we use cylindrical polar coordinates r, z, θ centred at the middle of the disc. Here

$$\begin{aligned}
B_r &= C\left[\frac{2z(1+r^2+z^2)}{r((1-r)^2+z^2)((1+r)^2+z^2)^{1/2}}E\left(2\left(\frac{r}{(1+r)^2+z^2}\right)^{1/2}\right)\right. \\
&\quad \left. -\frac{2z}{r((1+r)^2+z^2)^{1/2}}K\left(2\left(\frac{r}{(1+r)^2+z^2}\right)^{1/2}\right)\right], \\
B_z &= C\left[\frac{2}{((1+r)^2+z^2)^{1/2}}K\left(2\left(\frac{r}{(1+r)^2+z^2}\right)^{1/2}\right)\right. \\
&\quad \left. +\frac{2\left(1-r^2-z^2\right)}{((1-r)^2+z^2)((1+r)^2+z^2)^{1/2}}E\left(2\left(\frac{r}{(1+r)^2+z^2}\right)^{1/2}\right)\right],
\end{aligned}$$

where C gives the strength of the magnet and $K(.)$ is the complete elliptic integral of the first kind and $E(.)$ the the complete elliptic integral of the second kind as defined in [3]. It should be possible to work out C using the manufacturers calibrations of these magnets. However these give the field strength at the surface of the magnet, which is where greatest variations in the field occur, and so are completely useless. Before estimates of the force on magnetic particles can be obtained the magnets being used in experiments need to be properly calibrated and C evaluated.

In figure 1 a plot is made of the magnitude of the body force exerted on a superparamagentic for a particle of strength $\Upsilon \mathrm{Vol}/2\mu_0 = 1$ in the field surrounding a disc magnet.



(a) (b)

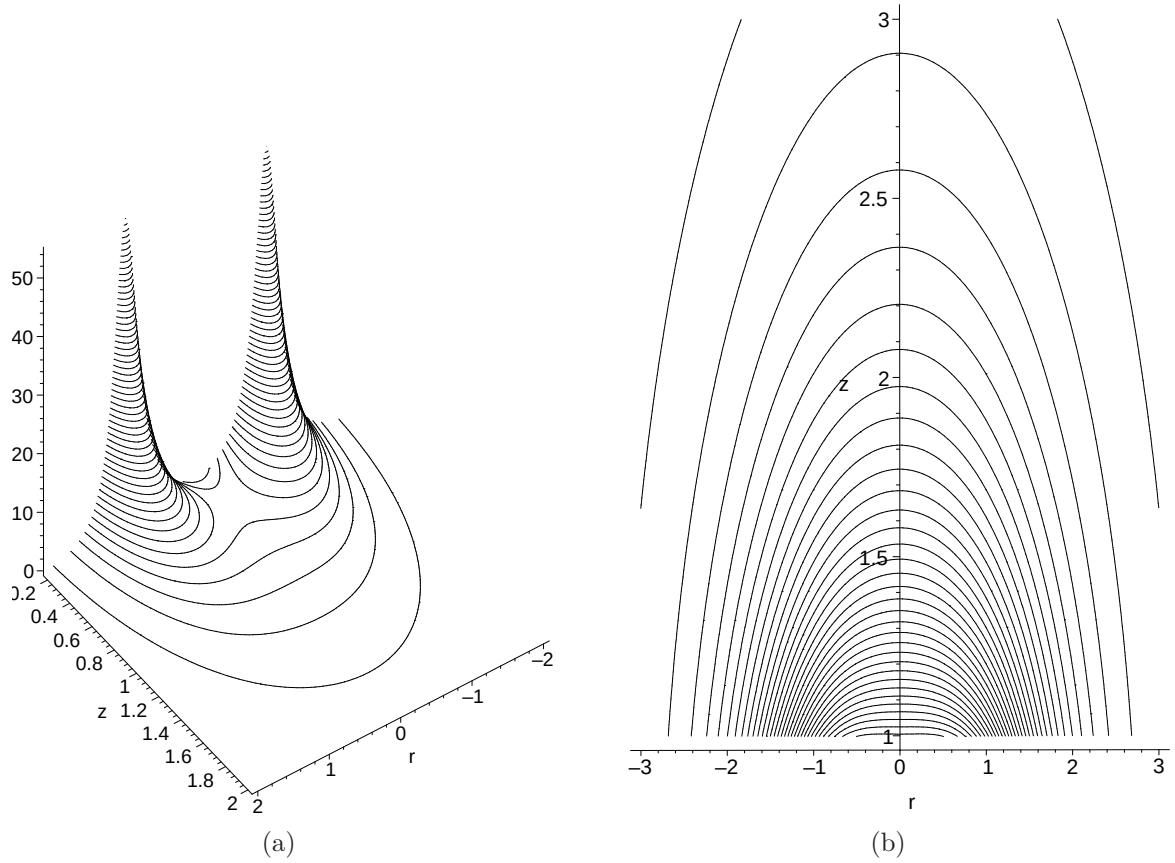
Figure 1: Plot of $|(\mathbf{B}\cdot\nabla)\mathbf{B}|$ for a disc magnet of unit radius with $C = 1$. a) 3D representation including field near magnet, b) contourplot.

A similar plot is made for a magnetic dipole strength $|\boldsymbol{m}| = 1$ in figure 5.2 assuming the dipole is aligned with the magnetic field. Finally a plot of the maximum torque exerted on a particle with $|\boldsymbol{m}| = 1$, in other words a plot of $|\boldsymbol{B}|$, is made in 3. In all cases we have also assumed that $C = 1$. The plots can be scaled linearly to obtain plots of forces for particles and magnets of different strengths from those chosen here.

5.3 Models for the mechanical properties of cells

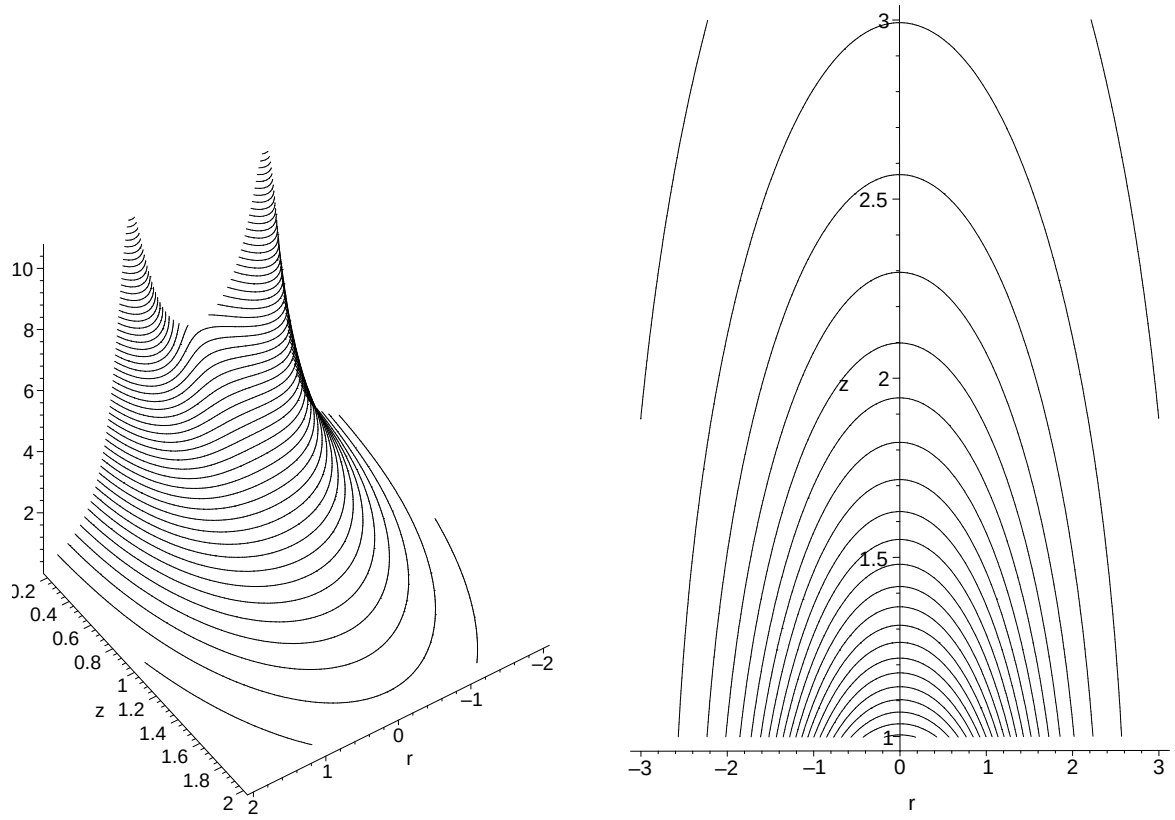
In this section we investigate the mechanical properties of a cell when it is subjected to a force from a magnetic particle attached to its surface. The size of the magnetic particles under consideration range from $10nm - 1\mu m$ while the typical lengthscale for a cell is $10\mu m$. For simplicity we shall only consider relatively small particles which have dimensions much less than the cell. This allows us to model the force acting on the surface of the cell as a point force.

The question now arises how to model the cell. The mechanical properties of many different types of cell have been investigated experimentally using atomic force microscopes, optical traps (laser tweezers) and micropipettes [1, 6]. In [1] the literature arising from micropipette studies has been reviewed. This article suggests (at least over short time-scales) that cells such as a chondrocyte behave as an elastic ball whereas cells such



(a) (b)

Figure 2: Plot of $\frac{|(\mathbf{B}\cdot\nabla)\mathbf{B}|}{|\mathbf{B}|}$ for a disc magnet of unit radius with $C = 1$. a) 3D representation including field near magnet, b) contourplot.



(a) (b)

Figure 3: Plot of $|\mathbf{B}|$ for a disc magnet of unit radius with $C = 1$. a) 3D representation including field near magnet, b) contourplot.

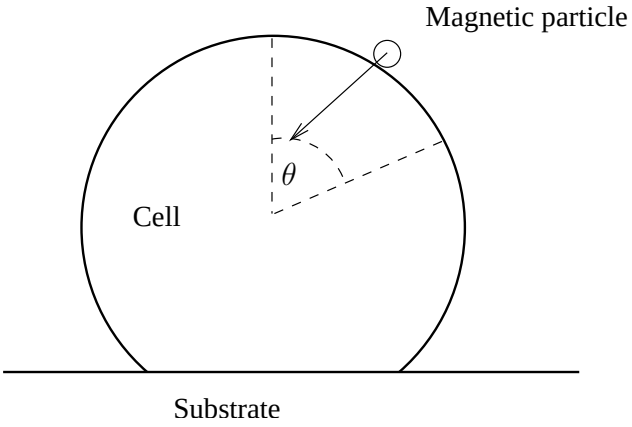


Figure 4: The configuration of the osteoblast.

as neutrophils behave as a fluid sac encased in an elastic membrane. There is less information available on osteoblasts but it is believed that they have behaviour analogous to neutrophils. It is worth stating that these continuum models do not meet with universal approval and that there is another school of thought, exemplified by Ingber [5], which prefers to model the cell as a complex tensegrity structure in which there are a mixture of compression and tension bearing elements. However, due to the impractical nature of constructing such a model (how many elements should be used? where should they be placed?), we opt for the continuum approach.

5.3.1 Model for an osteoblast

The model

We model the osteoblast as a fluid filled sac surrounded by an elastic shell (lipid membrane). In a healthy state the membrane is under tension and the cell has an approximately spherical shape. In practice these cells are usually bound to a substrate and so, where this is planar, we expect the osteoblast to take the form of a spherical cap (see figure **??**). The stress tensor for the membrane is two-dimensional and takes the form

$$\sigma_{ij} = (T_0 + K\varepsilon_{kk})\,\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{2}\varepsilon_{kk}\delta_{ij}\right),$$

where $\underline{\underline{\varepsilon}}$ is the strain tensor, T_0 is the membrane tension in the absence of applied forces, K is the dilatation modulus and μ is the shear modulus. Bending stiffness of the membrane is neglected as might be expected from an asymptotic analysis.

We assume that the cell is a spherical cap, with radius $R = L$, when the membrane is under tension T_0 but otherwise unstressed. Magnetic particles attached to the outside of the cell membrane exert a force per unit area $\boldsymbol{f}$ on the surface of the cell and the fluid inside the cell exerts a pressure p outwards on the membrane. These forces cause a deformation of the membrane $\boldsymbol{u}$. We now nondimensionalise the problem with the unstressed radius of the cell L and the typical force exerted by the magnetic particle F

and write

$$R = LR', \quad \boldsymbol{u} = \frac{F}{\mu}\boldsymbol{u}', \quad \boldsymbol{f} = \frac{F}{L^2}\boldsymbol{f}',$$
$$p = \frac{F}{L^2}p', \quad \sigma_{ij} = \frac{F}{L}\sigma'_{ij}, \quad \varepsilon_{ij} = \frac{F}{\mu L}\varepsilon'_{ij}$$

so that the stress tensor takes the form

$$\sigma_{ij} = \left(T^* + K^*\varepsilon_{kk}\right)\delta_{ij} + 2\left(\varepsilon_{ij} - \frac{1}{2}\varepsilon_{kk}\delta_{ij}\right), \tag{5.2}$$

where

$$T^* = \frac{LT}{F}, \qquad K^* = \frac{K}{\mu}. \tag{5.3}$$

Henceforth we shall drop the primes.

In these dimensionless units the physical components of the strain tensor take the form

$$\varepsilon_{\theta\theta} = (u_\theta + w), \tag{5.4}$$
$$\varepsilon_{\theta\phi} = \left(v_\theta + \frac{1}{\sin\theta}u_\phi - \cot\theta v\right), \tag{5.5}$$
$$\varepsilon_{\phi\phi} = \left(\frac{v_\phi}{\sin\theta} + \cot\theta u + w\right), \tag{5.6}$$

where we use spherical polar coordinates θ, ϕ on the surface of the sphere and the displacement of the membrane $\boldsymbol{u} = u(\theta,\phi)\boldsymbol{e}_\theta + v(\theta,\phi)\boldsymbol{e}_\theta + w(\theta,\phi)\boldsymbol{e}_r$. Writing the force per unit area $\boldsymbol{f}$ exerted by the magnetic particles in the form

$$\boldsymbol{f} = f_n(\theta,\phi)\boldsymbol{e}_r + f_t(\theta,\phi)\boldsymbol{e}_\theta + f_p(\theta,\phi)\boldsymbol{e}_\phi,$$

and balancing these forces with the pressure exerted by the fluid and the elastic stresses in the cell membrane, assuming small displacements, leads to the following formulae:

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\sigma_{\theta\theta}\right) + \frac{\partial}{\partial\phi}\sigma_{\theta\phi} - \cos\theta\sigma_{\phi\phi}\right) = f_t, \tag{5.7}$$
$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\sigma_{\theta\phi}\right) + \frac{\partial}{\partial\phi}\sigma_{\phi\phi} + \cos\theta\sigma_{\theta\phi}\right) = f_p, \tag{5.8}$$
$$(\sigma_{\theta\theta} + \sigma_{\phi\phi}) = p - f_n. \tag{5.9}$$

The first two equations represent tangential force balances in the membrane while the third represents a normal force balance perpendicular to the cell membrane. In addition the volume of the cell remains constant.

Assuming that the cell is bound onto a planar substrate on $\theta = b$ equations (5.2)-(5.9) should be solved together with the boundary condition

$$u = v = 0 \qquad \text{on } \theta = b, \tag{5.10}$$

and, since $\theta = 0$ is a singular point of the system (5.4)-(5.6) and (5.7)-(5.9), the boundary condition

$$u \text{ and } v \text{ are non-singular on } \theta = 0, \tag{5.11}$$

is also required. Note that we do not specify $w = 0$ on $\theta = b$ because we have neglected the bending stiffness of the membrane, and hence also, the second derivatives of w in the model. The apparent paradox can be resolved by considering a boundary layer region about $\theta = b$ in which bending stiffness is reintroduced to the model.

In addition to the equations above a relation between the pressure p and the volume of fluid contained within the cell must be specified. In this context it is reasonable to assume incompressibility of the fluid such that the displacement causes no change in the cell volume. For small displacements $u(\theta,\phi)\boldsymbol{e}_\theta + v(\theta,\phi)\boldsymbol{e}_\theta + w(\theta,\phi)\boldsymbol{e}_r$ and for a cell whose initial configuration is the spherical cap $r \in [0,1]$, $\theta \in [0,b]$, $\phi \in [0,2\pi)$ this gives rise to the relation

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{b}\int_{r=0}^{1+w(\theta,\phi)} \left(r^2\sin\theta + r(\sin\theta u)_\theta + rv_\phi\right) drd\theta d\phi - \int_{r=0}^{1}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{b} r^2\sin\theta drd\theta d\phi = 0.$$

Simplifying this further, assuming small displacements, we find that

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{b} w\sin\theta d\theta d\phi + \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{b} \left((\sin\theta u)_\theta + v_\phi\right) d\theta d\phi = 0,$$

Where we require that v is periodic in ϕ and that displacements at the pole ($\theta = 0$) are continuous this expression simplifies further to give

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{b} w\sin\theta d\theta d\phi = 0, \tag{5.12}$$

Equations for the displacement. Substituting (5.4)-(5.6) into (5.9) gives the following equations for w in terms of u and v:

$$w = \frac{1}{4K^*}(p - f_n - 2T^*) - \frac{1}{2}\left(u_\theta + \frac{v_\phi}{\sin\theta} + \cot\theta u\right). \tag{5.13}$$

We now substitute (5.4)-(5.6) into (5.7)-(5.8) and make use of (5.13) to eliminate w this leads to the following coupled ODEs for u and v:

$$\begin{aligned}\left(\sin^2\theta u_{\theta\theta} + 2u_{\phi\phi} + u_\theta\sin\theta\cos\theta - \cos(2\theta)u + \sin\theta v_{\theta\phi} - 3\cos\theta v_\phi\right)& \\ = \left(f_t - \frac{1}{2}\frac{\partial f_n}{\partial\theta}\right)\sin^2\theta,& \qquad (5.14)\\ \left(2\sin^2\theta v_{\theta\theta} + v_{\phi\phi} + \sin(2\theta)v_\theta - 2\cos(2\theta)v + \sin\theta u_{\theta\phi} + 3\cos\theta u_\phi\right)& \\ = \left(f_p - \frac{1}{2\sin\theta}\frac{\partial f_n}{\partial\phi}\right)\sin^2\theta.& \qquad (5.15)\end{aligned}$$

Given the forcing functions f_n, f_t and f_p it is possible to solve the model given by (5.10), (5.11), (5.12) and (5.13)-(5.15). In practice this can be accomplished by solving (5.14)-(5.15) together with the boundary conditions (5.10)-(5.11) to determine u and v, then substituting these expressions into (5.13) and finally using the volume constraint (5.12) to determine the unknown pressure p and hence also w.

An exact solution to the model We look to model the situation in which a magnetic particle sits on top of the spherical cap (at $\theta = 0$) and exerts unit force downward through the centre of the cell. In this scenario there is radial symmetry such that $v = 0$ and all ϕ derivative vanish. Thus (5.14)-(5.15) reduce to the single equation

$$\left(\sin^2\theta u_{\theta\theta} + u_\theta \sin\theta\cos\theta - \cos(2\theta)u\right) = \left(f_t - \frac{1}{2}\frac{\partial f_n}{\partial\theta}\right)\sin^2\theta.$$

On parts of the cell not in contact with the magnetic particle the right hand side of the above equation is zero and consequently has solution

$$u = \alpha\sin\theta + \beta\left(2\sin\theta\log\left(\tan(\theta/2)\right) - 2\cot\theta\right), \tag{5.16}$$

where α and β are constants of integration. Applying the boundary condition (5.10) gives

$$\alpha = 2\beta\left(\frac{\cos b}{\sin^2 b} - \log\left(\tan(b/2)\right)\right). \tag{5.17}$$

The radial displacement w follows from (5.13) and has the form

$$w = -\alpha\cos\theta - 2\beta\left(1 + \cos\theta\log\left(\frac{\sin\theta}{1+\cos\theta}\right)\right) + \frac{1}{4K^*}\left(p - 2T^*\right). \tag{5.18}$$

Calculating the stress tensor from constitutive tensor (5.2) we find

$$\sigma_{\theta\theta} = \frac{p}{2} + \frac{4\beta}{\sin^2\theta},$$
$$\sigma_{\phi\phi} = \frac{p}{2} - \frac{4\beta}{\sin^2\theta},$$

so that if we have a force of strength D acting vertically downwards in a small region about $\theta = 0$ it follows that

$$\beta = -\frac{D}{8\pi}. \tag{5.19}$$

Although the displacement at $\theta = 0$ is singular we can still use (5.12) to determine the pressure since the singularity can be regularised by either reintroducing the small bending stiffness of the membrane, and matching (5.16)-(5.17) to a small inner region in which bending stiffness is non-negligible, or, by matching to an inner region on the lengthscale of the particle in which a contact problem must be solved. Application of (5.12) to (5.16)-(5.18) gives

$$p = 2T^* + K^*\left(4\beta + \alpha\frac{1-\cos(2b)}{1-\cos b} + 4\beta\frac{\sin^2 b}{1-\cos b}\log\left(\frac{\sin b}{1+\cos b}\right)\right).$$

Substituting for α from (5.17) we obtain

$$p - 2T^* = \frac{2\beta K^*}{\sin^2 b/2} = -\frac{DK^*}{4\pi\sin^2 b/2}, \tag{5.20}$$

and hence

$$\begin{aligned}\sigma_{\theta\theta} &= T^* - \frac{DK^*}{8\pi\sin^2(b/2)} - \frac{2D}{2\pi\sin^2\theta},\\ \sigma_{\phi\phi} &= T^* - \frac{DK^*}{8\pi\sin^2(b/2)} + \frac{2D}{2\pi\sin^2\theta},\end{aligned} \tag{5.21}$$

where we have also used (5.19) to replace β.

Remarks on the validity of the model. The dimensional displacements of the membrane become comparable with the radius of the cell if $F/(L\mu) = O(1)$ and the small displacement theory breaks down. Estimates for the force exerted by a magnetic particle are $F = O(10pN)$, the radius of the cell $L = O(5\mu m)$ and, assuming the cell membrane is a lipid bilayer, for the dilatational modulus $K = O(0.5N/m)$ and shear modulus $\mu = O(10^{-5}N/m)$. This gives a rough estimate of $F/(L\mu) = 1/5$. There is, however, some doubt about the value of μ. Note also that the extra tension in the membrane is proportional to K^* and that, with these estimates, $K^* = O(5 \times 10^4)$. Our model, however, will almost certainly break down in the region immediately surroundingthe particle. This is because the diameter of the particle $l = 10nm - 1\mu m$ and so $F/(l\mu)$ will always become comparable with 1 in the vicinity of the particle where the solution to the model is singular. In order to rectify this we need to use a nonlinear elastic model in a small region surrounding the particle. We expect this to significantly alter our result. In particular note that the stress in the membrane has a component proportional to $K^* = K/\mu$ which is very large; this is because the membrane is unable to display significant inward deformation close to the particle (the small strain assumption). If large deformation were allowed it would be possible to support the particle using the tension of the membrane and hence one would expect the membrane stresses to be much smaller.

A possible method of numerical solution of the model. Since the above model is linear we can make a Fourier decomposition in ϕ to reduce the PDEs (5.14)-(5.15) to an infinite set of coupled ODEs. In the case where a point force $C\delta(\phi)\delta(\theta - a)\boldsymbol{e}_\theta/\sin a + D\delta(\phi)\delta(\theta - a)\boldsymbol{e}_r/\sin a$ is applied at the point $\theta = a$, $\phi = 0$, modelling the action of a magnetic particle, the decomposition proceeds as follows:

$$\sum_{n=0}^{\infty} \cos(n\phi)U_n(\theta), \quad \sum_{n=1}^{\infty} \sin(n\phi)V_n(\theta), \quad \sum_{n=0}^{\infty} \cos(n\phi)W_n(\theta).$$

Using the fact that the Fourier decomposition of the δ-function is

$$\sum_{m=-\infty}^{\infty} \delta(\phi - 2m\pi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty} \cos(n\phi),$$

we write the resulting set of coupled ODEs as follows:

$$\sin^2\theta U_0'' + \sin\theta\cos\theta U_0' - \cos 2\theta U_0 = \frac{1}{2\pi}\delta'(\theta - a), \tag{5.22}$$

$$\begin{aligned}\sin^2\theta U_n'' + \sin\theta\cos\theta U_n' - (\cos(2\theta) + 2n^2)U_n + n\sin\theta V_n' - 3n\cos\theta V_n \\ = \frac{1}{\pi}\left(\frac{C}{\sin a}\delta(\theta - a) - \frac{D}{2\sin a}\delta'(\theta - a)\right)\sin^2\theta,\end{aligned} \tag{5.23}$$

$$\begin{aligned}2\sin^2\theta V_n'' + \sin(2\theta)V_n' - (2\cos(2\theta) + n^2)V_n - n\sin\theta U_n' - 3n\cos\theta U_n \\ = \frac{nD}{2\pi\sin a}\delta(\theta - a)\sin\theta.\end{aligned} \tag{5.24}$$

The resulting equations for U_n and V_n, where $n \geq 1$, have the asymptotic behaviour

$$U_n \sim M_1\theta^{1+n} + M_2\theta^{1-n} + M_3\theta^{-1-n} + M_4\frac{n-4}{n+2}\theta^{n-1},$$

$$V_n \sim M_1\theta^{1+n} - M_2\theta^{1-n} + \frac{2-n}{n+4}M_3\theta^{-1-n} + M_4\theta^{n-1},$$

about $\theta = 0$ where M_1, M_2, M_3 and M_4 are arbitrary constants. Whereas equation (5.22) has solutions with asymptotic behaviour

$$U_0 \sim L_1\theta + L_2\theta^{-1}.$$

Hence, in order that u and v are non-singular at $\theta = 0$, we require

$$M_2 = M_3 = L_2 = 0.$$

We also require

$$U_n = V_n = 0, \qquad \text{on} \quad \theta = a.$$

Equations (5.22)-(5.25) give a well-posed boundary value problem for the U_n and V_n although it is probably easier to deal with the homogenous version of (5.22)-(5.24) and impose the jump conditions

$$[U_0]_a = -\frac{D}{4\pi \sin a}, \quad [U_0']_a = -\frac{D}{2\pi \sin a}, \quad [U_n]_a = -\frac{D}{2\pi \sin a},$$
$$[U_n']_a = -\frac{D}{\pi \sin a}, \qquad [V_n]_a = 0, \qquad [V_n']_a = \frac{n}{4\pi \sin^2 a}.$$

Once this system has been solved the radial displacement w and the pressure can be calculated from (5.13) and (5.12). In practice the jumps in U_n give rise to Gibb's phenomena which suggests that a smoothing technique should be used in conjuction with the Fourier decomposition.

This method is easily generalised to the case where a component of the point force is in the $\boldsymbol{e}_\phi$ direction. All that is needed is to carry out a general Fourier decomposition of U and V.

5.3.2 A model for chondrocytes

In line with [1] we model the chondrocyte as a solid elastic ball. The stress tensor takes the form

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right),$$

where K is bulk modulus and μ is the shear modulus of the cell. Since the chondrocyte is approximately spherical in an unstressed state it is again appropriate to use spherical polar coordinates (r, θ, ϕ) and write the displacements in the form $\boldsymbol{u} = u(\theta,\phi)\boldsymbol{e}_\theta + v(\theta,\phi)\boldsymbol{e}_\theta + w(\theta,\phi)\boldsymbol{e}_r$. The physical components of the strain tensor can then be found on p.7 of the Landau and Lifshitz volume on elasticity [4]. The equations satisfied by the displacement are then

$$\frac{3\mu}{3K+\mu}\nabla^2\boldsymbol{u} + \boldsymbol{\nabla}(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0},$$

and should be solved in conjuction with the boundary conditions:

$$\underline{\underline{\sigma}}\cdot\boldsymbol{e}_r\big|_{r=R} = \boldsymbol{F}(\theta,\phi),$$

(here R is the radius of the unstressed cell and $\boldsymbol{F}(\theta,\phi)$ is the force per unit area applied on the surface of the cell) and

$$u = v = w = 0 \quad \text{on } r\cos\theta = R\cos a.$$

5.3.3 Further work

With regard to the model for the osteoblast the most pressing need is to formulate a nonlinear elastic model to describe the membrane mechanics close to the magnetic particle. We expect this to significantly alter the results obtained from the linear theory and, in particular, we expect the stresses in the membrane to be significantly smaller once this alteration to the model has been made.

In practice chondrocytes exhibit viscous relaxation over a long timescales [1]. This suggests that the model given in (5.3.2) should be replaced by a visco-elastic model if we wish to study this longer timescale behaviour. We also expect something similar may occur for the osteoblast, not only because the membrane may exhibit visco-elastic behaviour, but also because the excess pressure $p - 2T_0$ generated by the deformation will almost certainly give rise to a flow of fluid from the cell to its surrounds.

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